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INSTRUMENTAL VARIABLES ESTIMATION AND THE COEFFICIENT OF DETERMINATION

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We first demonstrate that Instrumental Variables (IV) may be viewed as a two - step OLS estimator. Subsequently, we show that the coefficient of determination based on IV residuals does not lie between 0 and 1. As an alternative we suggest the use of the coefficient of determination from the second step of the two - step OLS interpretation of IV. The properties of said R^2 are investigated and its relationship to the R^2 suggested by Carter and Nagar for simultaneous equations is examined.

1. Introduction

The coefficient of determination (R^2) is a popularly used statistic by applied econometricians. This fact alone warrants showing extreme care in using the proper R^2 for the particular statistical model utilised. For example, for the linear regression model without an intercept term which has been estimated by Ordinary Least Squares (OLS), the usual R^2 defined in terms of the mean deviation form of the data is inappropriate. Such a statistic would not lie between zero and unity (Aigner (1971: 85-90)). The appropriate R^2 would be the one defined in terms of raw data (i.e., data not in mean-deviation form).

Our concern here is with a similar problem. We shall consider estimating a single linear equation by Instrumental Variables (IV) and shall suggest an R^2 statistic which may be properly utilised under such conditions. With this in mind, we shall **first**

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demonstrate that IV estimation may be viewed as a two-step OLS procedure. This point is definitely not original; it would not be far from the truth to admit that, for the present author, it was inspired by the Two-Stage Least Squares (2SLS) estimation method utilised to estimate a single linear equation within a system of simultaneous equations. In the literature, this result was obtained with a totally different motivation by Sargan (1958); Liviatan (1963) and Wallis (1967) suggested its use within the context of distributed lag estimation. Finally, a demonstration in relation to the Limited Information Iterated Instrumental Variables (LIIV) method is given by Dhrymes (1971 : 174-175).

We shall **next** show that the usual R^2 obtained by using IV residuals is not the appropriate statistic to use since it does not lie between 0 and 1 but between $-\infty$ and 1. We shall **then** suggest using the coefficient of determination from the second step of the two-step OLS interpretation of IV. The properties of said R^2 will be investigated and compared with the R^2 statistic suggested by Carter and Nagar (1977) for simultaneous equations.

2. The Model

We have the following single regression equation,

$$y = X\beta + u \tag{1}$$

where y is Tx1, X is TxK, β is Kx1 and u is Tx1. Letting k denote the number of explanatory variables, $K = k+1$ if (1) contains an intercept term and $K = k$ if it does not.

The equation in (1) may constitute a single-equation model implying a unilateral "causality" between the explanatory variables and the dependent variable or it may be an identified member of a system of simultaneous equations, interdependently related to the other equations in the system. In either case, it will be governed by the following set of assumptions :

A.1 The vector of disturbances u has zero mean and variance-covariance matrix $\sigma^2 I_K$.

A.2 The matrix of observations on the explanatory variables, namely X, is correlated with the vector of disturbances u, so that plim $T^{-1}X'u \neq 0$.

A.3 Let there exist a TxK matrix of instrumental variables Z such that,

a. the matrices Z'Z and Z'X both have rank K,

b. plim $T^{-1}Z'Z = M_{zz}$ and plim $T^{-1}Z'X = M_{zx}$ are finite and nonsingular,

c. plim $T^{-1}Z'u = 0$,

d. $T^{-1/2} Z'u$ converges in distribution to $N(0, \sigma^2 M_{zz})$.

Assumption A.2 implies that at least one of the columns of X is correlated with the elements of the disturbance vector u. We know that this constitutes a violation of the basic assumptions of the linear regression model and renders the OLS estimator of β biased and inconsistent.

Assumption A.3 indicates that the instrumental variables satisfy the properties of being uncorrelated with the disturbance term (A.3.c) and being correlated with the explanatory variables (A.3.b). It further requires that $T^{-1/2} Z'u$ have a well-defined asymptotic distribution (A.3.d). This last assumption is necessary as the first three assumptions in A.3 may, in certain cases, not be sufficient to establish the asymptotic distribution of the IV estimator of β. (For a detailed discussion and a case in point, see Schmidt (1976 : 102-105).)

3. IV as Two-Step OLS

The IV principle seeks to provide us with estimators of β which are at least consistent under A.2. The IV estimator is usually derived in the following way.

Let $\hat{u}$ be a vector of residuals obtained by using any estimator of β, say $\hat{\beta}$; i.e., $\hat{u} = y - X\hat{\beta}$. If $\hat{\beta}$ is obtained by solving the following set of K equations;

(2) $$Z'\hat{u} = Z'y - Z'X\hat{\beta} = 0$$

then it would be the IV estimator of β, namely $\hat{\beta}_{IV}$ and, by A.3.a, would be calculated as,

(3) $$\hat{\beta}_{IV} = (Z'X)^{-1} Z'y$$

The asymptotic distribution of $\hat{\beta}_{IV}$ may be proved, under A.3 (b,c,d), to be

(4) $N(\beta, \sigma^2 M_{zx}^{-1} M_{zz} M_{xz}^{-1})$

(see Schmidt (1976 : 101-102)) where the covariance matrix would be consistently estimated by

(5) $\hat{\sigma}^2 (Z'X)^{-1} Z'Z (X'Z)^{-1}$

and where

(6) $\hat{\sigma}^2 = (y - X\hat{\beta}_{IV})'(y - X\hat{\beta}_{IV})/T$

Now consider the following two-step procedure :

Step 1 : Regress, by OLS, the columns of X on Z to obtain the fitted values $\tilde{X} = Z(Z'Z)^{-1}Z'X$.

Step 2 : Regress, again by OLS, y on $\tilde{X}$ to obtain

(7) $\tilde{\beta} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'y = (X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y$

First, note that, by A.3.a,

(8) $\tilde{\beta} = (Z'X)^{-1}Z'Z\,(X'Z)^{-1}X'Z\,(Z'Z)^{-1}Z'y = (Z'X)^{-1}Z'y = \hat{\beta}_{IV}$

i.e., $\tilde{\beta}$ is identical to the IV estimator of β.

Secondly, since $\tilde{\beta} = \beta + (\tilde{X}'\tilde{X})^{-1}\tilde{X}'u$, we may express,

(9) $T^{1/2}(\tilde{\beta} - \beta) = (T^{-1}\tilde{X}'\tilde{X})^{-1}T^{-1/2}\tilde{X}'u$

Hence, if we can show that, (a) plim $T^{-1}\tilde{X}'\tilde{X}$ is finite and nonsingular, and (b) $T^{-1/2}\tilde{X}'u$ has a well-defined asymptotic distribution, we may then establish the consistency of $\tilde{\beta}$ and the asymptotic distribution of $T^{1/2}(\tilde{\beta} - \beta)$.

To establish point (a) we take the probability limit of $T^{-1}\tilde{X}'\tilde{X}$;

(10) $$plim\ T^{-1}\tilde{X}'\tilde{X} = plim\ (T^{-1}X'Z(T^{-}Z'Z)^{-1}T^{-1}Z'X) = M_{XZ}M_{ZZ}^{-1}M_{ZX}$$

by A.3.b and, of course, is nonsingular by the same assumption.

To establish point (b), we note that

(11) $$T^{-1/2}\tilde{X}'u = T^{-1}X'Z(T^{-1}Z'Z)^{-1}T^{-1/2}Z'u$$

and, since by A.3b and d, plim $T^{-1}X'Z = M_{xz}$, plim $T^{-1}Z'Z = M_{zz}$ and $T^{-1/2}Z'u$ is asymptotically normal with mean zero and co-variance matrix $\sigma^2 M_{zz}$, it follows that, asymptotically,

(12) $$T^{-/2}\tilde{X}'u \sim N(O,\ \sigma^2 M_{XZ}M_{ZZ}^{-1}M_{ZX})$$

We may, then, conclude that (1) plim $T^{-1}\tilde{X}'u = 0$ so that

(13) $$plim\ \tilde{\beta} = \beta + (plim\ T^{-}\tilde{X}'\tilde{X})^{-1}\ plim\ T^{-1}\tilde{X}'u = \beta$$

i.e., $\tilde{\beta}$ is consistent, and (2) that asymptotically,

(14) $$T^{1/2}(\tilde{\beta} - \beta) \sim N(O,\ \sigma^2(M_{XZ}M_{ZZ}^{-1}M_{ZX})^{-1})$$

But note that, by A.3.b,

(15) $$\sigma^2(M_{XZ}M^{-1}M_{ZX})^{-1} = \sigma^2 M_{ZX}^{-1}M_{ZZ}M_{XZ}^{-1}$$

so that $\tilde{\beta}$ and $\hat{\beta}_{IV}$ have the same asymptotic distribution. We have thus proved,

Theorem 1: The two-step OLS estimator $\tilde{\beta}$;

i. is identical to $\hat{\beta}_{IV}$,

ii. is consistent and asymptotically distributed as $N(\beta,\ \sigma^2(M_{xz}M_{zz}^{-1}M_{zx})^{-1})$, which is identical to the asymptotic distribution of $\hat{\beta}_{IV}$.

In proving the consistency of the two-step estimator, nothing is mentioned about the consistency of the estimators in the first step regression. Letting C denote the KxK matrix of coefficients from this regression, we have $\tilde{X} = Z\tilde{C}$ where $\tilde{C} = (Z'Z)^{-1}Z'X$. Since the consistency of the two - step estimator requires that plim $T^{-1}\tilde{X}'u = 0$, we take an alternative look at plim $T^{-1}\tilde{X}'u$;

$$(16) \qquad plim\ T^{-1}\tilde{X}'u = (plim\ \tilde{C}')\ (plim\ T^{-1}Z'u)$$

Since plim $T^{-1}Z'u=0$ by A.3.c, all that is needed for plim $T^{-1}\tilde{X}'u=0$ to hold is that plim $\tilde{C} = A < \infty$. A need not equal C; i.e., $\tilde{C}$ need not be a consistent estimator of C. We have thus proved,

Corollary 1.1 : The consistency of the two-step OLS version of IV estimation does not require the coefficient estimators of the first step regression to be consistent.

In the discussion above we said nothing about how the matrix of instruments may be formed. In fact, the columns of the matrix Z may consist of direct observations on the variables which act as instrumental variables or they may be formed by some estimation or fitting procedure. The latter situation may arise when there are more than enough directly observed variables which may act as instrumental variables. Denoting the matrix of all such likely candidates by Z^* which is TxK^* and $K^* \geq K$, any TxK submatrix of Z^* would qualify as the Z matrix but, in each case, a certain amount of information would not be utilised in consistenlty estimating β. **One way**[1] of making use of all the columns of Z^* is to regress the columns of X on Z^*, obtain the matrix of fitted values, say $\tilde{X}^*$, and use $\tilde{X}^*$ as the matrix of instruments. $\tilde{X}^*$ obviously satisfies the requirements of an IV matrix since its columns consist of linear combinations of the columns of Z^* which are uncorrelated with u, and since it is obtained by maximising the multiple correlation between each column of X and the columns of Z^*.

(1) Another way of obtaining fitted values, in the context of simultaneous equations, is to use the estimated restricted reduced form, as in the case of the LIIV estimator mentioned above. The ensuing conclusions will not be applicable in this case, however, since the fitted values so obtained will not be orthogonal to the resultant residuals (see Dhrymes (1971 : 171)).

The IV estimator now becomes,

$$(17) \qquad \hat{\beta}_{IV} = (\tilde{X}^{*\prime}X)^{-1}\tilde{X}^{*\prime}y$$
$$= (X'Z^*(Z^{*\prime}Z^*)^{-1}Z^{*\prime}X)^{-1}X'Z^*(Z^{*\prime}Z^*)^{-1}Z^{*\prime}y$$
$$= (\tilde{X}^{*\prime}\tilde{X}^*)^{-1}\tilde{X}^{*\prime}y$$

The last equality in (17) indicates that the present $\hat{\beta}_{IV}$ is also a two-step OLS estimator. That it gives results identical to that of the two-step OLS precedure described above is easy to demonstrate. Within the context of the two-step procedure described previously, the first step would consist of regressing X on $\tilde{X}^*$ to obtain

$$(18) \qquad \tilde{X}^{**} = \tilde{X}^* (\tilde{X}^{*\prime}\tilde{X})^{-1}\tilde{X}^{*\prime}X$$

But $\tilde{X}^{*\prime}\tilde{X}^* = \tilde{X}^{*\prime}X$ so that $\tilde{X}^{**} = \tilde{X}^*$, and the first step of both two-step procedures yield identical results. Since the second step is the same in both instances, we have demonstrated our contention.

We may thus conclude;

Corollary 1.2 : If the matrix of instruments Z consist only of directly observable variables, then the two-step procedure of Theorem 1 will still yield IV estimators even if the coulmn dimension of Z is greater than K.

Remark 1 : We have, so far, couched our discussion in terms of all the explanatory variables in equation (1) even though, in interpreting A.2 we pointed out that the correlation between X and u may only hold for a single column of X. This implies that if only a subset of the columns of X are correlated with u, then the remaining columns may act as their own instruments. Hence, partitioning X as $X = (X_1, X_2)$ where X_i is TxK_i, $i = 1,2$ and $K_1+K_2=K$, the instrument matrix may be defined as $Z = (Z_1, X_2)$ where Z_1 is TxK_1^* and $K_1^* \geq K_1$. To see that Theorem 1, Corrolaries 1.1 and 1.2 still hold in this case, all we need to show is that, in the first step of the two-step procedure, the predicted value of K_2 is identical to itself. For this purpose define $S_2 = (0', I_{K_2})$, where 0 is $K_1^*xK_2$, and express X_2 as $X_2 = ZS_2$. Then,

(19) $$\tilde{X}_2 = Z\ (Z'Z)^{-1}Z'X_2 = Z\ (Z'Z)^{-1}Z'ZS_2 = ZS_2 = X_2$$

Thus, in the second step only the fitted values of X_1 need be used; X_2 remains unaltered.[2]

Remark 2 : We shall, for future reference consider the relationship between the IV residuals û and the residuals from the second step regression, say $\tilde{v} = y - \tilde{X}\tilde{\beta}$. Now, û may be expressed as

(20) $$\hat{u} = y - X\hat{\beta}_{IV} = (I - X\ (Z'X)^{-1}Z')y = (I - X(\tilde{X}'\tilde{X})^{-1}\tilde{X}')y$$

and $\tilde{v}$ as,

(21) $$\tilde{v} = y - \tilde{X}\tilde{\beta} = (I - \tilde{X}\ (\tilde{X}'\tilde{X})^{-1}\tilde{X}')y$$

But $\tilde{X} = X - \tilde{W}$ where $\tilde{W}$ is the matrix of residuals from the first stage regression of X on Z, so that

(22) $$\tilde{v} = (I - X\ (\tilde{X}'\tilde{X})^{-1}\tilde{X}' + \tilde{W}\ (\tilde{X}'\tilde{X})^{-1}\tilde{X}')y$$

$$= \hat{u} + \tilde{W}\hat{\beta}_{IV}$$

4. R^2 in IV Estimation

We now turn to considering the coefficient of determination within the context of IV estimation.

The concept of a coefficient of determination, be it in terms of raw data or in terms of data in mean-deviation form, requires that the total sum of squares of the dependent variable be decomposed into two disjoint components; one corresponding to the fitted part and the other to the residual part. In any case, where this decomposition cannot be obtained, a coefficient of

(2) We should hasten to point out that if $K_1^* > K_1$ and, instead of regressing X on Z to form instruments we choose to select a subset from Z which has the same column dimension as X, then this selection must strictly be made from the K_1^* observed instruments not from the set of all K* observed instruments. In other words, if a set of explanatory variables act as their own instruments then they must always be included in the Z matrix used in the first step regression; otherwise $\tilde{\beta}$ (or $\tilde{\beta}_{IV}$) will not be consistent. A formal demonstration is provided in the next footnote.

determination will not be well-defined since it will not lie between 0 and 1. Hence, our **first** task here will be to demonstrate that such a decomposition cannot be obtained for an IV estimator; therefore the usual R^2 statistic based on the IV residuals $\hat{u}$ will not be well-defined.

Assume either that the linear relation in (1) does not contain an intercept term or, if it does, that the data has been put into mean-deviation form so that y and X now represent the matrices for the transformed data. In any event, we shall express the total sum of squares as y'y and write it explicitly as,

$$y'y = \hat{y}'\hat{y} + \hat{u}'\hat{u} + 2\,\hat{y}'\hat{u} \tag{23}$$

For y'y to be expressible only as the sum of $\hat{y}'\hat{y}$ and $\hat{u}'\hat{u}$, the cross-term $\hat{y}'\hat{u}$ must vanish. But.

$$\hat{y}'\hat{u} = \hat{\beta}_{IV}'X'\hat{u} \tag{24}$$

and $X'\hat{u} \neq 0$ since the estimating equations in (2) do not require $X'\hat{u} = 0$ to hold but for $Z'\hat{u} = 0$ to hold. Thus, the required decomposition is not obtained.

If one insists on using the conventional definitions of R^2 one may obtain two different R^2 measures from the decomposition in (23), namely

$$R_1^2 = 1 - (\hat{u}'\hat{u}/y'y) \text{ and } R_2^2 = \hat{y}'\hat{y}/y'y \tag{25}$$

where the relationship between the two is

$$R_1^2 = R_2^2 + (2\hat{y}'\hat{u}/y'y) \tag{26}$$

If one obtains a perfect fit so that $\hat{u} = 0$, then $R_1^2 = 1$. In other words, R_1^2 has an upper bound of unity. Furthermore, since R_2^2 is always positive, R_1^2 may well be negative whenever $2\hat{y}'\hat{u} < 0$ and $|\,2\hat{y}'\hat{u}\,| > \hat{y}'\hat{y}$.

We now turn to our **second** task of suggesting an R^2 statistic which is well-defined. Turning once again to the decomposition in (23), we find that since $X = \tilde{X} + \tilde{W}$,

(27) $\hat{y}'\hat{y} = \hat{\beta}_{IV}'X'X\hat{\beta}_{IV} = \hat{\beta}_{IV}'\tilde{X}'\tilde{X}\hat{\beta}_{IV} + \hat{\beta}_{IV}'\tilde{W}'\tilde{W}\hat{\beta}_{IV}$

and by (22),

(28) $\hat{u}'\hat{u} = \tilde{v}'\tilde{v} + \hat{\beta}_{IV}'\tilde{W}'\tilde{W}\hat{\beta}_{IV} - 2\hat{\beta}_{IV}'\tilde{W}'\tilde{v}$

(29) $2\hat{y}'\hat{u} = 2(\hat{\beta}_{IV}'X'\tilde{v} - \hat{\beta}_{IV}'X'\tilde{W}\hat{\beta}_{IV})$

$= 2(\hat{\beta}_{IV}'\tilde{W}'\tilde{v} - \hat{\beta}_{IV}'\tilde{W}'\tilde{W}\hat{\beta}_{IV})$

which upon substituting in (23) yields,

(30) $y'y = \hat{\beta}_{IV}'\tilde{X}'\tilde{X}\hat{\beta}_{IV} + \tilde{v}'\tilde{v}$

This is nothing but the decomposition one would obtain from the second step regression, and the resultant R^2 would be

(31) $R^2_{IV} = \hat{\beta}_{IV}'\tilde{X}'\tilde{X}\hat{\beta}_{IV}/y'y = 1 - (\tilde{v}'\tilde{v}/'y'y)$

The relationships between R_{IV}^2 and, R_1^2 and R_2^2 may be obtained as

(32) $R_{IV}^2 = (\hat{y}'\hat{y}/y'y) - (\hat{\beta}_{IV}'\tilde{W}'\tilde{W}\hat{\beta}_{IV}/y'y)$

$= R_2^2 - (\hat{\beta}_{IV}'\tilde{W}'\tilde{W}\hat{\beta}_{IV}/y'y)$

and

(33) $R_{IV}^2 = R_1^2 - ((\hat{\beta}_{IV}'\tilde{W}'\tilde{W}\hat{\beta}_{IV} + 2\hat{\beta}_{IV}'\hat{y}'\hat{u})/y'y)$

$= R_1^2 - ((\hat{u}'\hat{u} - \tilde{v}'\tilde{v})/y'y)$

Hence, $R_{IV}^2 < R_2^2$, and $R_{IV}^2 \lesseqgtr R_1^2$ depending upon whether $\tilde{u}'\tilde{u} \gtreqless \tilde{v}'\tilde{v}$.

The justification we have so far given for the use of R_{IV}^2 is that it lies between 0 and 1. On the other hand, one may very well object that this R^2 measure does not really measure the goodness-of-fit of the model in (1) to the data, but that of

(34) $$y = \tilde{X}\beta + u + \tilde{W}\beta$$

i.e., the induced regression equation of the second step.[3] Hence, we must provide justifications for its use that go beyond the fact that it lies in the [0,1] interval. For this purpose we should distinguish between the cases where $\tilde{C}$ is not a consistent estimator and where it is a consistent estimator.

In the **first** case, the argument for using R_{IV}^2 as the goodness-of-fit measure in IV estimation would be based on how close $\tilde{W}$, the matrix of residuals from the first step regression would be to the zero matrix. If $\tilde{X}$ is regarded as a **proxy** for X, then the better the proxy the smaller would be the difference between R_{IV}^2 and R_1^2 and R_2^2; i.e., the better would any R^2 statistic lying between 0 and 1 approximate an R^2 statistic based on the actual IV residuals $\hat{u}$. Thus, in this case we are regarding R_{IV}^2 as a purely descriptive device and argue that its reliability will be higher depending upon how well the instruments in the first step are chosen.[4]

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(3) The point made in the previous footnote may be formally demonstrated in terms of (34) above. We have.

$$\hat{\beta}_{IV} = \beta + (\tilde{X}'\tilde{X})^{-1}\tilde{X}'u + (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{W}\beta = \beta + (\tilde{X}'\tilde{X})^{-1}\tilde{X}'u$$

since $\tilde{X}'\tilde{W} = 0$. Now, let $\tilde{X}' = (\tilde{X}_1, X_2)$. Then

$(\tilde{X}'\tilde{W})' = ((\tilde{X}_1'\tilde{W})', (X_2'\tilde{W})')$

But if $X_2 = (X_2^*, X_2^{**})$ where X^* consists of the disturbance - uncorrelated explanatory variables in equation (1) included in the first step regression and X_2^{**} consists of those which are left out, then

$$\tilde{W}'X_2 = (\tilde{W}'X_2^*, \tilde{W}'X_2^{**}) = (O, \tilde{W}'X_2^{**}) \neq O$$

and $\tilde{X}'\tilde{W} \neq 0$, so that

$$\hat{\beta}_{IV} - \beta \neq (\tilde{X}'\tilde{X})^{-1}\tilde{X}'u$$

and our proof of consistency on p. 365 above will no longer hold.

(4) One can, of course, make use of the all-purpose goodness-of-fit statistic suggested by Haessel (1978) which involves obtaining the correlation coefficient between y and $\hat{y} = X\hat{\beta}_{IV}$ as an alternative descriptive statistic. The only problem with this statistic would be the difficulty of establishing a simple relation between it and the IV residuals $\hat{u}$.

In the second case, the consistency of $\tilde{C}$ implies that the first step regression involves a well-defined model as

(35) $$X = ZC + W = \bar{X} + W$$

where it is assumed that (i) E (W) = 0, (ii) E (W'W) = Ω which is positive definite, (iii) E(W'u) = w ≠ 0, and (iv) E(Z'W) = 0. Consequently, the model in the second step regression may now be written as

(36) $$y = \bar{X} \beta + v$$

where v = u + W β. If we now investigate the probability limit of the total sum of squares y'y we would obtain,

(37) $$plim\ T^{-1}y'y = \beta' (plim\ T^{-1}\bar{X}'\bar{X}) \beta + plim\ T^{-1}v'v + 2\ \beta'\ plim\ \bar{X}'v$$

But,

(38) $$plim\ T^{-1}\bar{X}'\bar{X} = C'M_{ZZ}C$$

by A.3.b;

(39) $$\begin{aligned} plim\ T^{-1}v'v &= plim\ T^{-1}u'u + \beta' (plim\ T^{-1}W'W) \beta + 2\ \beta'\ plim\ T^{-1}W'u \\ &= \sigma^2 + \beta'\Omega\ \beta + 2\ \beta'\ w \\ &= E(v'v) \end{aligned}$$

and

(40) $$\begin{aligned} plim\ T^{-1}\bar{X}'v &= C' plim\ T^{-1}Z'v \\ &= C' (plim\ T^{-1}Z'u + plim\ T^{-1}Z'W \beta) \\ &= C'0 = 0 \end{aligned}$$

by A.3.c, and by (ii), (iii) and (iv) above. Hence, we may define a population coefficient of determination as,

$$(41) \qquad \rho^2 = \beta' C' M_{ZZ} C\beta / (plim\ T^{-1} y'y)$$
$$= \beta' C' M_{ZZ} C\beta / (\beta' C' M_{ZZ} C\beta + E(v'v))$$

Now, R_{IV}^2 would be a consistent estimator of ρ^2 if $\tilde{C}$ is a consistent estimator of C, since in that case

$$(42) \qquad plim\ T^{-1}\tilde{X}'\tilde{X} = (plim\ \tilde{C}')\ (plim\ T^{-1}Z'Z)\ (plim\ \tilde{C})$$
$$= C' M_{ZZ} C = plim\ T^{-1}\bar{X}'\bar{X}$$

and

$$(43) \qquad plim\ T^{-1}\tilde{v}'\tilde{v} = plim\ T^{-1}v'v - (plim\ T^{-1}v'\tilde{X})\ (plim\ T^{-1}\tilde{X}'\tilde{X})^{-1}$$
$$.(plim\ T^{-1}\tilde{X}'v)$$
$$= plim\ T^{-1}v'v - O.(C' M_{ZZ} C)^{-1}.O$$
$$= plim\ T^{-1}v'v = E(v'v)$$

because

$$(44) \qquad plim\ T^{-1}\tilde{X}'v = (plim\ \tilde{C}')\ (plim\ T^{-1}Z'u + plim\ T^{-1}Z'W\beta) = 0.$$

We have thus established,

Theorem 2: Given the models in (35) and (36), (i) ρ^2 as given in (41) is a population coefficient of determination for (36), and (ii) if C is consistently estimated and β is estimated by $\hat{\beta}_{IV}$, then R_{IV}^2 is a consistent estimator of ρ^2.

The significance of R_{IV}^2 being a consistent estimator of ρ^2 lies in the fact that it may be utilised in testing the hypothesis

$H_o : \rho^2 = 0$

which is the same thing as testing

$H_o : \beta = 0$

since $\rho^2 = 0$ if and only if $\beta = 0$. We may use the following test statistic for this purpose;

(45) $$q = \hat{\beta}_{IV}' \tilde{X}' \tilde{X} \hat{\beta}_{IV} / (\tilde{v}' \tilde{v} / T) = TR_{IV}^2 / (1 - R_{IV}^2)$$

Following the reasoning in Carter and Nagar (1977 : 43) we may show that under H_o, $T^{1/2} \hat{\beta}_{IV}$ converges in distribution to $N(0, \sigma^2 C' M_{zz} C)$ so that q converges in distribution to a χ^2 variable with k degrees of freedom.

Remark 3: Carter and Nagar (1977) have developed a coefficient of determination (henceforth denoted by R_{CN}^2) for single equation estimation in simultaneous equation systems. Since almost all single equation estimation methods may be interpreted as IV estimators (see Klein (1955) for 2SLS, Goldberger (1965) for the general k-class, and Brundy and Jorgenson (1971) for other IV estimators), the R_{IV}^2 statistic becomes a natural rival to the R_{CN}^2 statistic.

Let us give a brief description of R_{CN}^2. In terms of the notation utilised in this paper, X may be regarded as the matrix of explanatory variables in the structural equation considered so that (1) becomes that particular equation, and Z may be taken to be the matrix of all exogeneous variables in the system so that (35) may be regarded as the reduced form for X[5] and (36) as the partially restricted reduced form for y. The R_{CN}^2 is then defined to be

(46) $$R_{CN}^2 = \tilde{\beta}' \tilde{X}' \tilde{X} \tilde{\beta} / (\tilde{\beta}' \tilde{X}' \tilde{X} \tilde{\beta} + \tilde{v}' \tilde{v})$$

where $\tilde{\beta}$ is a consistent, single equation estimator of the structural coefficients, $\tilde{X} = Z\tilde{C}$ where $\tilde{C}$ is **any** consistent estimator of the reduced form coefficients, and $\tilde{v} = y - \tilde{X}\tilde{\beta}$. Note that if $\tilde{\beta}$ is given by 2SLS then R_{IV}^2 and R_{CN}^2 become identical. For any other consistent estimator of C the two statistics differ. This is because in the case of R_{CN}^2, Z always consists of all the exogeneous variables in the system while in the case of R_{IV}^2, Z may contain instruments which have been previously obtained by a fitting procedure as in LIIV.

(5) Strictly speaking the reduced form of relevance is for the submatrix of X corresponding to the explanatory current endogeneous variables. But in view of Remark 1, using (35) as the reduced form does not change the analysis.

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Ö Z E T

ARAÇ DEĞİŞKENLERLE TAHMİN VE BELİRLEME KATSAYISI

Bu çalışmada ilkin Araç Değişkenler tahmin yönteminin iki aşamalı en küçük kareler (2AEKK) yöntemi olarak yorumlanabileceği gösterilmektedir. İkinci olarak, Araç Değişkenler yöntemiyle tahmin edilen bir denklemden elde edilen artıklara dayanılarak hesaplanan belirleme katsayısının (R^2) 0 ile 1 arasında yer almadığı gösterilmekte ve bunun yerine, 2AEKK yorumu bağlamında, ikinci aşamada elde edilen R^2'nin kullanılması önerilmektedir. Üçüncü olarak, söz konusu istatistiğin özellikleri incelenmekte ve Carter - Nagar tarafından eşçözümlü denklem modelleri için önerilen R^2 istatistiği ile olan ilişkisi saptanmaktadır.